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# Representation of Value Imperfection with the Aid of Background Knowledge: H-IFS

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**Summary.** An Integrated Data Management System is required for representing and managing indicative information from multiple sources describing the state of an enterprise. In such environments, information may be partially known because the related information from the real world corresponds to a set of possible values including the unknown. Here, we present a way to replace unknown values using background knowledge of data that is often available arising from a concept hierarchy, as integrity constraints, from database integration, or from knowledge possessed by domain experts. We present and examine the case of H-IFS to represent support contained in subsets of the domain as a candidate for replacing unknown values mostly referred in the literature as NULL values.

## 1 Introduction

Background knowledge of data is often available, arising from a concept hierarchy, as integrity constraints, from database integration, or from knowledge possessed by domain experts. Frequently integrated DBMSs contain incomplete data which we may represent by using H-IFS to declare support contained in subsets of the domain. These subsets may be represented in the database as partial values, which are derived from background knowledge using conceptual modelling to re-engineer the integrated DBMS. For example, we may know that the value of the attribute JOB-DESCRIPTION is unknown for the tuple relating to employee Natalie but also know from the attribute salary that Natalie receives an estimated salary in the range of  $\text{€}5\text{K} \sim \text{Salary}_{25\text{K}}$ . A logic program, using a declarative language can then derive the result that Natalie is a “Junior- Staff”, which we input to the attribute JOB-DESCRIPTION of tuple Natalie in the re-engineered database. In such a manner we may use the knowledge base to replace much of the unknown in the integrated database environment.

Generalised relations have been proposed to provide ways of storing and retrieving data. Data may be imprecise, hence we are not certain about the

specific value of an attribute but only that it takes a value which is a member of a set of possible values. An extended relational model for assigning data to sets has been proposed by [1]. This approach may be used either to answer queries for decision making or for the extraction of patterns and knowledge discovery from relational databases. It is therefore important that appropriate functionality is provided for database systems to handle such information.

A model, which is based on partial values [2], has been proposed to handle imprecise data. Partial values may be thought of as a generalisation of null values where, rather than not knowing anything about a particular attribute value, we may identify the attribute as a set of possible values. A partial value is therefore a set such that exactly one of the values in the set is the true value.

We review the different types of NULL values and then we focus is on providing an integrated DBMS environment that will enable us to:

- Reconcile unknown information with the aid of background knowledge. We examine the appropriateness of the H-IFS as a form of Background knowledge for replacing unknown attribute values
- Utilise constraints as part of the integrated DBMS metadata to improve query execution
- Query imprecise information a part of integrated DBMS environment that may entail more than one sources of information

## 2 Review of NULL Values

A null value represents an *unknown* attribute value, is a value that is known to exist, but the actual value is unknown. The unknown value is assumed to be a valid attribute value, that is, some value in the domain of that attribute. This is a very common kind of ignorant information. For example, in an employee database, while everyone must have a surname, Alex's surname may be recorded as unknown. The unknown value indicates that Alex has a name, but we do not know her name. An unknown value has various names in the literature including *unknown null* [3], *missing null* [4], and *existential null* [5].

The meaning of a fact,  $F$ , with an unknown attribute value over an attribute domain of cardinality  $N$  is a multiset with  $N$  members; each member is a set containing an  $F$  instance with the unknown value being replaced by a different value from the attribute domain. For example, assume  $f = \{\text{IBM}(\perp)\}$  where  $(\perp)$  represents an unknown value over a domain  $\{20, 22\}$  with respect to IBM's share-price), then the meaning of  $f$  is

$$f = \{\{\text{IBM}(20)\}, \{\text{IBM}(22)\}\}$$

This corresponds to the notion that a fact with an unknown value is incomplete with respect to a fact where that unknown value is no longer unknown, but is now known to be a specific value (i.e.  $f_1 = \{\{\text{IBM}(20)\}\}$ ).

Another generalization of an unknown fact is a *disjunctive* fact [6], so known as *indefinite* information [7]. A disjunction is a logical *or* applied to fact instances. Let  $F$  be an inclusive disjunctive fact with  $N$  disjuncts. The meaning of  $F$  is given by a multiset with  $N$  members; each member is a set containing one disjunct. For example, the share price of IBM may be £20 or £22. (i.e. “IBM (20), IBM (22)”). The disjunction is *exclusive* [8] or *inclusive* [9]. If it is an exclusive disjunction, one and only one disjunct is true. The meaning of an exclusive disjunctive fact is the same as that of an imprecise value. Let  $f = \{\{\text{IBM}(20)\}, \{\text{IBM}(22)\}\}$  be an exclusive disjunctive then the meaning of  $f$  is  $f = \{\text{IBM}(20)\} \vee \{\text{IBM}(22)\}$ .

The meaning of an inclusive disjunctive fact is somewhat different than that of its exclusive complement, at least one alternative may be true. Let  $F$  be an inclusive disjunctive fact with  $N$  disjuncts. The meaning of  $F$  is given by a multiset with  $2^N - 1$  members; each member is a unique subset of disjuncts. For example, assume, the inclusive disjunct  $f = \{\text{IBM}(20)\{\text{IBM}(22)\}$  then the meaning of  $f$ , is  $f = \{\{\text{IBM}(20)\}, \{\text{IBM}(22)\}, \{\text{IBM}(20), \text{IBM}(22)\}\}$ , excluding the fact,  $\{\{\text{IBM}(\perp)\}\}$ . The empty ( $\perp$ ) attribute represents the situation where a fact instance exists, but does not have a particular attribute-label value.

A *maybe value* is an attribute-label value which might or might not exist [10]. If it does exist, the value is known. A *maybe tuple or fact-instance* is similar to a maybe value, but the entire tuple might not be part of the relation. Maybe tuples are produced when one disjunct of an inclusive disjunctive fact-instance is found to be true.

A combination of inclusive disjunctive fact instance and a maybe fact instance can determine the semantics of *open information* or nulls [11]. The denotation of an open null is exact to inclusive disjunctive information with the addition of the empty set as a possible value. That is the attribute-label value may not exist, could be exactly one value, or could be many values. For example, in the shares database an open value could be used to present IBM share prices. This value means that IBM share price possibly had a past record, (this could be the first appearance in the market); IBM share price may be one or many. The open value covers all this possibilities. A generalization of open information is *possible* information [12] (this differs from our use of the term “possible”). Possible information is an attribute value whose existence is undetermined, but if it does exist, it could be multiple values from a *subset* of the attribute domain.

A *no information* value is a combination of an open value and an unknown value [13]. The no information value restricts an open value to resemble an unknown value. A, no information value might not exist, but if it does, then it is a single value, which is unknown, rather than possibly many values. The meaning of a no information value is similar to that of an *unknown* value with the inclusion of the addition of the empty set as a possible value.

Unknown, partially known, open, no information, and maybe null values are different interpretations of a null value. There are other null value interpretations, but none of these is a kind of well cognisant information.

An *inapplicable* or *does not exist* null is a very common null value. An inapplicable null, appearing as an attribute value, means that an attribute does not have a value [14]. An inapplicable value neither contains nor represents any ignorance; it is known that the attribute value does not exist. Inapplicable values indicate that the schema (usually for reasons of efficiency or clarity) does not adequately model the data. The relation containing the inapplicable value can always be decomposed into an equivalent set of relations that do not contain it. Hence the presence of inapplicable values indicates inadequacies in the schema, but does not imply that information is being incompletely encoded.

Open nulls is the main representative of the possible-unweighted-unrestricted branch. Universal nulls may also be classified under this branch assuming the OWA semantics. Inclusive disjunctive information, possible information and maybe tuples or values are indicative representatives of the possible-unweighted-restricted school.

In [15] five different types of nulls are suggested. The labels and semantics of them are defined as follows. Let  $V$  be a function, which takes a label and returns a set of possible values that the label may have.

Intuitively,  $V(\text{Ex-mar}) = D$  says that the actual value of an existential marker can be any member of the domain  $D$ . Likewise,  $V(\text{Ma-mar}) = D \cup \{\perp\}$  says that the actual value of a maybe marker can be either any member of  $D$ , or the symbol  $\perp$ , denoting a non-existent value. Similarly,  $V(\text{Par-mar}(V_s)) = V_s$  says that the actual value of a partial null marker of the form  $pa\ mar\ (V_s)$  lies in the set  $V_s$ , a subset of the domain  $D$ .

An important issue is the use of  $\perp$ , which denotes that an attribute is inapplicable. However such an interpretation of the unknown information, is not consistent with the principles of conceptual modelling. Assuming the sample fact spouse, the individual, Tony, is a bachelor and hence, the wife field is inapplicable to him,  $\perp$ . Conceptually the issue can be resolved with the use of the subtypes (e.g. married, unmarried) as part of the entity class Person. A subtype is introduced only when there is at least one role recorded for that subtype. The conceptual treatment of null will permit us to reduce the table in Fig. 1 using only two types of null markers (Fig. 2).

Label ( $\chi$ )	$V(\chi)$
Ex-mar	$D$
Ma-mar	$D \cup \{\perp\}$
Pl-mar	$\{\perp\}$
Par-mar ( $V_s$ )	$V_s$
Pm-mar ( $V_s$ )	$V_s \cup \{\perp\}$

Fig. 1. Types of NULL and their semantics

Label ( $\chi$ )	$V(\chi)$
V-mar ( $V$ )	$\{V\}$
P-mar ( $V_s$ )	$\{V_s\}$
$\Pi$ -mar ( $D-V_s$ )	$\{D-V_s\}$

**Fig. 2.** The reduced set of NULL values

In the general case the algebraic issue under the use of subtypes is whether the population of the subtypes in relationship to the super type is:

- *Total and disjoint:* Populations are mutually exclusive and collectively exhaustive.
- *Non-total and disjoint:* Populations are mutually exclusive but not exhaustive.
- *Total and overlapping:* Common members between subtypes and collectively exhaustive, in relationship to super type.
- *Non-total and overlapping:* Common members between subtypes and not collectively exhaustive, in relationship to super type.

Conclusively it can be said that a null value is often semantically overloaded to mean either an unknown value or an inapplicable. For an extensive coverage of the issues related to the semantic overloading of null values somebody may further refer to [16].

### 3 NULL Values and Background Knowledge in DBMS

In a generalised relational database we consider an attribute  $A$  and a tuple  $t_i$  of a relation  $R$  in which  $n$  attribute value  $t_i[A]$  may be a partial value. A partial value is formally defined as follows.

**Definition 3.1.** *A partial value is determined by a set of possible attribute values of tuple  $t$  of attribute  $A$  of which one and only one is the true value. We denote a partial value by  $P = [a_1, \dots, a_n]$  corresponding to a set of  $P$  possible values  $\{a_1, \dots, a_n\}$  of the same domain, in which exactly one of these values is the true value of. Here,  $P$  is the cardinality of  $\{a_1, \dots, a_n\}$  is a subset of the domain set  $\{a_0, \dots, a_{n+1}\}$  of attribute  $A$  of relation  $R$ , and  $P \leq n + 1$ .*

Queries may require operations to be performed on partial values; this can result in a query being answered by means of bags, where the tuples have partial attribute values [17].

*Example 3.1.* We consider the attribute JOB.DESCRPTION that has possible values ‘Research Associate’, ‘Teaching Associate’, ‘Programmer’, ‘Junior Staff’, and ‘Senior Staff’. Then  $\{\text{‘Junior Staff’}, \text{‘Senior Staff’}\}$ , is an example of a partial value in terms of classical logic. In this case we know only that the individual is a staff but not whether he or she is a junior staff or a senior staff.

**Definition 3.2.** An Intuitionistic Fuzzy partial value relation  $R$ , is a relation based on partial values for domains  $D_1, D_2, \dots, D_n$  of attributes  $A_1, A_2, \dots, A_n$  where  $R \subseteq P_1 \times P_2 \times \dots \times P_n$  and  $P_i$  is the set of all the partial values on power set of domain  $D_i$ . A pruned value of attribute  $A_i$  of the relation  $R$  corresponds then to a H-IFS which is a subset of the domain  $D_i$ . An example of a partial value relation is presented in Table 1 below:

Let  $l$  be an element defined by a structured domain  $D_i$ .  $U(e)$  is the set of higher level concepts, i.e.  $U(e) = \{n | n \in D_i \wedge n \text{ is an ancestor of } l\}$ , and  $L(e)$  is the set of lower concepts  $L(e) = \{n | n \in D_i \wedge n \text{ is a descendent of } l\}$ . If  $l$  is a base concept then  $L(e) = \emptyset$  and if  $l$  is a top level concept, then  $U(e) = \emptyset$ . Considering the H-IFS  $F = \{\text{Research Associate}/\langle 1.0, 0.0 \rangle, \text{Programmer}/\langle 0.7, 0.2 \rangle, \text{Teaching Associate}/\langle 0.4, 0.1 \rangle\}$  as part of the Concept Employee in Fig. 3:

$$U(\text{Programmer}/\langle 0.7, 0.2 \rangle) = \{\text{Technical Staff}/\langle 0.7, 0.1 \rangle\}$$

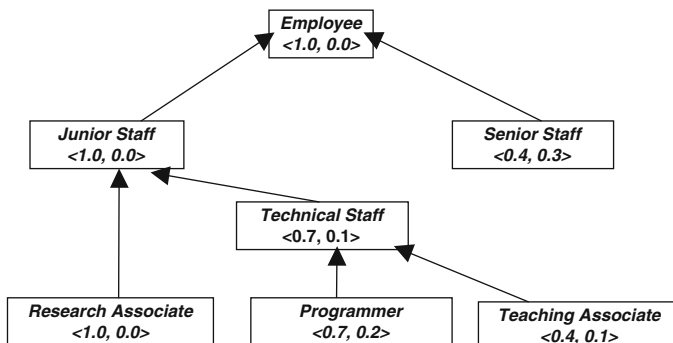
$$L(\text{Programmer}/\langle 0.7, 0.2 \rangle) = \emptyset$$

*Rule-1:* If  $(|U(e)| > 1 \wedge L(e) = \emptyset)$ , then it is simply declared that a child or base concept has many parents.

E.g.  $|U(\text{Programmer}/\langle 0.7, 0.2 \rangle)| = 3$ ,  $L(\text{Programmer}/\langle 0.7, 0.2 \rangle) = \emptyset$ , Therefore a child or base concept acting as a selection predicate can claim any tuple (parent) containing elements found in  $U(e)$ , as its ancestor.

**Table 1.** Generalised Relation Staff with partial values in the form of hierarchical IFS

Name	JOB_DESCRIPTION	SALARY
Natalie	{Research Associate/ $\langle 1.0, 0.0 \rangle$ }	{ $\sim \text{Salary}_{25K}$ }
Anna	{Programmer/ $\langle 0.7, 0.2 \rangle$ }	{ $\sim \text{Salary}_{20K}$ }



**Fig. 3.** Generalised H-IFS  $F$  – concept employee

Now let us consider the following case where  $l_1$  = “Employee” and  $l_2$  = “Programmer” then let  $B$  the function that defines the space between  $l_1 \wedge l_2$ . In this case  $|B((l_1), (l_2))| = 2$ , and  $> 1$

$B((l_1), (l_2)) = U(L(l_1) \wedge (l_2))$  where  $l_1$  is a high level concept,  $l_2$  is a base concept are elements defined in a structured domain. If both arguments are high level concepts or low level concepts then  $B((l_1), (l_2)) = \phi$ .

*Rule-2:* If  $B((l_1), (l_2))$  is defined and  $|B((l_1), (l_2))| > 1$ , then it is simply declared that multiple parents, high level concepts, are receiving a base concept as their own child. Therefore a parent or high level concept acting as a selection predicate can claim any tuple (child) containing elements found in  $(L(l_1) \wedge (l_2))$ , as its descendant, but with variants level of certainty.

Background knowledge may be specified as arising from a hierarchical Intuitionistic Fuzzy hierarchy, as integrity constraints, from the integration of conflicting databases, or from knowledge selected by domain experts. Using such information we offer to re-engineer the database by replacing missing, conflicting or unacceptable data by sets of the attribute domain.

Concept hierarchies have previously been used for attribute-induced knowledge discovery [18]. However the proposed use of background knowledge in this context is unique.

We assume that original attribute values may be given either as singleton sets, or subsets of the domain, or as concepts, which correspond to subsets of an attribute domain. In the last case the values may be defined in terms of a concept hierarchy. In addition there are rules describing the domain, and these may be formed in a number of ways: they may take the form of integrity constraints, where we have certain restrictions on domain values; functional dependencies and also rules specified by a domain expert.

An example of a concept hierarchy expressed with the aid of H-IFS  $F = \{\text{Research Associate}/\langle 1.0, 0.0 \rangle, \text{Programmer}/\langle 0.7, 0.2 \rangle, \text{Teaching Associate}/\langle 0.4, 0.1 \rangle\}$  with values which are sets given in the generalised relation staff in Table 1. Here a value for the attribute JOB\_DESCRIPTION may be a may be a concept from the concept hierarchy as defined in Fig. 3 (e.g.  $\{\text{Technical Staff}/\langle 0.7, 0.1 \rangle\}$ ). Then in terms of functional dependencies we may receive the following information  $\text{Technical-Staff} \rightarrow \sim \text{Salary}_{25K}$ . To this extent in terms of any declarative query language it can be concluded that the salary in for a reaching associate or a Programmer must be in the range of  $\text{Salary}_{25K}$ . We can also use this knowledge to approximate the salary for all instances of Junior Staff in case where no further background knowledge after estimating firstly the  $\langle \mu, \nu \rangle$  degrees for the hierarchical concept Employee. Such a hierarchical concept like employee in Fig. 3 can be defined with the aid of Intuitionistic fuzzy set over a universe [19, 20] that has a hierarchical structure, named as H-IFS.

## 4 Definition of IFS and H-IFS

### 4.1 Principles of Intuitionistic Fuzzy Sets – Atanassov’s Sets

Each element of an Intuitionistic fuzzy [21, 22] set has degrees of membership or truth ( $\mu$ ) and non-membership or falsity ( $\nu$ ), which do not sum up to 1.0 thus leaving a degree of hesitation margin ( $\pi$ ).

As opposed to the classical definition of a fuzzy set given by  $A' = \{ \langle x, \mu_{A'}(x) \rangle \mid x \in X \}$  where  $\mu_A(x) \in [0, 1]$  is the membership function of the fuzzy set  $A'$ , an Intuitionistic fuzzy set  $A$  is given by

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$$

where:  $\mu_A : X \rightarrow [0, 1]$  and  $\nu_A : X \rightarrow [0, 1]$  such that  $0 < \mu_A(x) + \nu_A(x) < 1$  and  $\mu_A(x), \nu_A(x) \in [0, 1]$  denote a degree of membership and a degree of non-membership of  $x \in A$ , respectively.

Obviously, each fuzzy set may be represented by the following Intuitionistic fuzzy set

$$A = \{ \langle x, \mu'_A(x), (x), 1 - \mu'_A(x) \rangle \mid x \in X \}$$

For each Intuitionistic fuzzy set in  $X$ , we will call  $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$  an Intuitionistic fuzzy index (or a hesitation margin) of  $x \in A$  which expresses a lack of knowledge of whether  $x$  belongs to  $A$  or not. For each  $x \in A$   $0 < \pi_A(x) < 1$ .

**Definition 4.1.1.** Let  $A$  and  $B$  be two fuzzy sets defined on a domain  $X$ .  $A$  is included in  $B$  (denoted  $A \subseteq B$ ) if and only if their membership functions and non-membership functions satisfy the condition:  $(\forall x \in X)(\mu_A(x) \leq \mu_B(x) \ \& \ \nu_A(x) \geq \nu_B(x))$

Two scalar measures are classically used in classical fuzzy pattern matching to evaluate the compatibility between an ill-known datum and a flexible query, known as:

- A possibility degree of matching,  $\Pi(Q; D)$
- A necessity degree of matching,  $N(Q/D)$

**Definition 4.1.2.** Let  $Q$  and  $D$  be two fuzzy sets defined on a domain  $X$  and representing, respectively, a flexible query and an ill-known datum:

- The possibility degree of matching between  $Q$  and  $D$ , denoted  $\Pi(Q; D)$ , is an “optimistic” degree of overlapping that measures the maximum compatibility between  $Q$  and  $D$ , and is defined by:

$$\Pi(Q/D) = \sup_{x \in X} \min(\langle 1 - \nu_Q(x), \nu_D(x) \rangle, \langle 1 - \nu_D(x), \nu_Q(x) \rangle)$$

- The necessity degree of matching between  $Q$  and  $D$ , denoted  $N(Q; D)$ , is a “pessimistic” degree of inclusion that estimates the extent to which it is certain that  $D$  is compatible with  $Q$ , and is defined by:

$$N(Q/D) = \inf_{x \in X} \max(\langle \mu_Q(x), 1 - \mu_Q(x) \rangle, \langle \mu_D(x), 1 - \mu_D(x) \rangle)$$



## 4.2 H-IFS

The notion of hierarchical fuzzy set rose from our need to express fuzzy values in the case where these values are part of taxonomies as for food products or microorganisms for example.

The definition domains of the hierarchical fuzzy sets that we propose below are subsets of hierarchies composed of elements partially ordered by the “kind of” relation. An element  $l_i$  is more general than an element  $l_j$  (denoted  $l_i \sim l_j$ ), if  $l_i$  is a predecessor of  $l_j$  in the partial order induced by the “kind of” relation of the hierarchy. An example of such a hierarchy is given in Fig. 1. A hierarchical fuzzy set is then defined as follows.

**Definition 4.2.1.** *A H-IFS is an Intuitionistic fuzzy set whose definition domain is a subset of the elements of a finite hierarchy partially ordered by the “kind of”  $\leq$  relation.*

*For example, the fuzzy set  $M$  defined as:  $\{Milk \langle 0.8, 0.1 \rangle, Whole-Milk \langle 0.7, 0.1 \rangle, Condensed-Milk \langle 0.4, 0.3 \rangle\}$  conforms to Definition-3. Their definition domains are subsets of the hierarchy given in Fig. 4.*

We can note that no restriction has been imposed concerning the elements that compose the definition domain of a H-IFS. In particular, the user may associate a given  $\langle \mu, \nu \rangle$  with an element  $l_i$  and another degree  $\langle \mu_1, \nu_1 \rangle$  with an element  $l_j$  more specific than  $l_i$ .  $\langle \mu, \nu \rangle \sim \langle \mu_1, \nu_1 \rangle$  represents a semantic of restriction for  $l_j$  compared to  $l_i$ , whereas  $\langle \mu_1, \nu_1 \rangle \sim \langle \mu, \nu \rangle$  represents a semantic of reinforcement for  $l_j$  compared to  $l_i$ . For example, if there is particular interest in condensed milk because the user studies the properties of low fat products, but also wants to retrieve complementary information about other kinds of milk, these preferences can be expressed using, for instance, the following Intuitionistic fuzzy set:  $\langle 1, 0 \rangle / \text{condensed milk} + \langle 0.5, 0.1 \rangle / \text{Milk}$ .

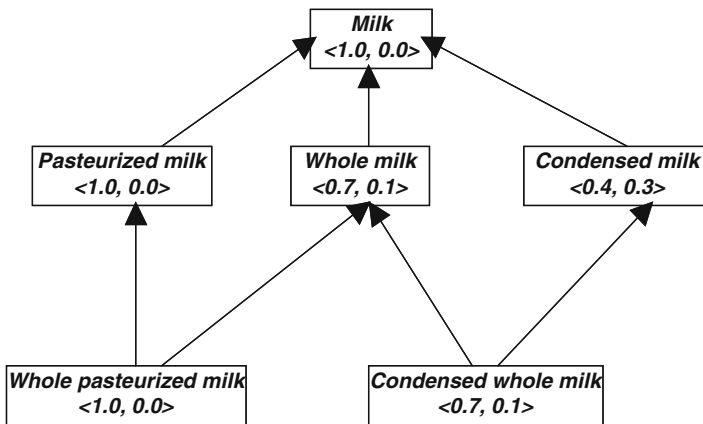


Fig. 4. Common closure of the H-IFSs  $Q$  and  $R$

In this example, the element condensed milk has a greater degree than the more general element Milk, which corresponds to a semantic of reinforcement for condensed milk compared to Milk.

### 4.3 Closure of the H-IFS

We can make two observations concerning the use of H-IFSs:

- Let  $\langle 1, 0 \rangle / \text{condensed milk} + \langle 0.5, 0.1 \rangle / \text{Milk}$  be an expression of liking in a query. We can note that this H-IFS implicitly gives information about elements of the hierarchy other than Condensed milk and Milk. One may also assume that any kind of condensed milk (i.e. whole condensed milk) interests the user with  $\langle \mu, \nu \rangle \diamond \langle 1, 0 \rangle$ .
- Two different H-IFSs on the same hierarchy do not necessarily have the same definition domain, which means they cannot be compared using the classic comparison operations of Intuitionistic fuzzy set theory “see Sect. 4.1”. For example,  $\langle 1, 0 \rangle / \text{condensed milk} + \langle 0.5, 0.1 \rangle / \text{Milk}$  and  $1 / \text{Milk} + 0.2 / \text{Pasteurised milk}$  are defined on two different subsets of the hierarchy of “Fig. 1” and, thus, are not comparable.

These observations led to the introduction of the concept of closure of a Intuitionistic hierarchical fuzzy set, which is defined on the whole hierarchy. Intuitively, in the closure of a H-IFS, the “kind of,  $\leq$ ” relation is taken into account by propagating the  $\langle \mu, \nu \rangle$  associated with an element to its sub-elements (more specific elements) in the hierarchy. For instance, in a query, if the user is interested in the element Milk, we consider that all kinds of Milk—Whole milk, Pasteurised milk, are also of interest. On the opposite, we consider that the super-elements (more general elements) of Milk in the hierarchy are too broad to be relevant for the user’s query.

**Definition 4.3.1.** Let  $F$  be a H-IFS defined on a subset  $D$  of the elements of a hierarchy  $L$ . Its degree is denoted as  $\langle \mu, \nu \rangle$ . The closure of  $F$ , denoted  $\text{clos}(F)$ , is a H-IFS defined on the whole set of elements of  $L$  and its degree  $\langle \mu, \nu \rangle_{\text{clos}(F)}$  is defined as follows.

For each element  $l$  of  $L$ , let  $S_L = \{l_1, \dots, l_n\}$  be the set of the smallest super-elements of  $l$  in  $D$ :

- If  $S_L$  is not empty,  $\langle \mu, \nu \rangle_{\text{clos}(F)}(S_L) = \langle \max_{1 \leq i \leq n} (\mu(L_i)), \min_{1 \leq i \leq n} (\nu(L_i)) \rangle$  else,  $\langle \mu, \nu \rangle_{\text{clos}(F)}(S_L) = \langle 0, 0 \rangle$

In other words, the closure of a H-IFS  $F$  is built according to the following rules. For each element  $l_1$  of  $L$ :

- If  $l_I$  belongs to  $F$ , then  $l_I$  keeps the same degree in the closure of  $F$  (case where  $S_L = \{l_I\}$ ).
- If  $l_I$  has a unique smallest super-element  $l_1$  in  $F$ , then the degree associated with  $l_I$  is propagated to  $l_1$  in the closure of  $F$ ,  $S_L = \{l_1\}$  with  $l_1 \succ l_I$

- If  $L$  has several smallest super-elements  $\{l_1, \dots, l_n\}$  in  $F$ , with different degrees, a choice has to be made concerning the degree that will be associated with  $l_I$  in the closure. The proposition put forward in Definition 4.3.1, consists of choosing the maximum degree of validity  $\mu$  and minimum degree of non validity  $\nu$  associated with  $\{l_1, \dots, l_n\}$ .
- All the other elements of  $L$ , i.e., those that are more general than, or not comparable with the elements of  $F$ , are considered as non-relevant. The degree  $\langle 0, 0 \rangle$  is associated with them.

Let us consider once more the H-IFS  $M$  defined as:  $\{\text{Milk } \langle 0.8, 0.1 \rangle, \text{Whole-Milk } \langle 0.7, 0.1 \rangle, \text{Condensed-Milk } \langle 0.4, 0.3 \rangle\}$  which is presented in Fig. 1.

The case of whole condensed milk is different: The user has associated the degree  $\langle 0.8, 0.1 \rangle$  with Milk, but has given a restriction on the more specific element whole milk (degree  $\langle 0.7, 0.1 \rangle$ ). As whole condensed milk is a kind of whole milk it inherits the  $\langle \mu, \nu \rangle$  associated with whole milk, that is  $\langle 0.7, 0.1 \rangle$ .

If the H-IFS expresses preferences in a query, the choice of the maximum allows us not to exclude any possible answers. In real cases, the lack of answers to a query generally makes this choice preferable, because it consists of enlarging the query rather than restricting it.

If the H-IFS represents an ill formulated concept, the choice of the maximum allows us to preserve all the possible values of the datum, but it also makes the datum less specific. This solution is chosen in order to homogenize the treatment of queries and data. In a way, it enlarges the query, answer.

#### 4.4 Properties of H-IFS

In Sect. 4.3 we saw that each H-IFS has an associated closure that is defined on the whole hierarchy.

We focus on the fact that two different H-IFSs, defined on the same hierarchy, can have the same closure, as in the following example.

*Example .* The H-IFSs  $Q = \{\text{Milk } \langle 1, 0 \rangle, \text{Whole-Milk } \langle 0.7, 0.1 \rangle, \text{Pasteurised-milk } \langle 1, 0 \rangle, \text{Condensed-Milk } \langle 0.4, 0.3 \rangle\}$  and  $R = \{\text{Milk } \langle 1, 0 \rangle, \text{Whole-milk } \langle 0.7, 0.1 \rangle, \text{Pasteurised-milk } \langle 1, 0 \rangle, \text{Whole-Pasteurised-milk } \langle 1, 0 \rangle, \text{Condensed Milk } \langle 0.4, 0.3 \rangle\}$  have the same closure, represented Fig. 4 below.

Such H-IFSs form equivalence classes with respect to their closures.

**Definition 4.4.1.** *Two H-IFSs  $Q$  and  $R$ , defined on the same hierarchy, are said to be equivalent  $Q \equiv R$  if and only if they have the same closure.*

**Property.** Let  $Q$  and  $R$  be two equivalent Intuitionistic hierarchical fuzzy sets. If  $l_I \in \text{dom}(Q) \cap \text{dom}(R)$ , then  $\langle \mu, \nu \rangle(Q.l_I) = \langle \mu, \nu \rangle(R.l_I)$

*Proof.* According to the definition of the closure of a H-IFS  $F$ , Definition 4.3.1, the closure of  $F$  preserves the degrees that are specified in  $F$ . As  $Q$  and  $R$  have the same closure (by definition of the equivalence), an element that belongs to  $Q$  and  $R$  necessarily has the same degree  $\langle \mu, \nu \rangle$  in both.

We can note that  $R$  contains the same element as  $Q$  with the same  $\langle \mu, \nu \rangle$ , and also one more element Whole-Pasteurised-milk  $\langle 1, 0 \rangle$ . The  $\langle \mu, \nu \rangle$  associated with this additional element is the same as in the closure of  $Q$ . Then it can be said that the element, Whole-Pasteurised-milk  $\langle 1, 0 \rangle$  is derivable in  $R$  through  $Q$ .

The same conclusions can be drawn in the case of condensed whole milk  $\langle 0.7, 0.1 \rangle$   $\square$

**Definition 4.3.2.** Let  $F$  be a hierarchical fuzzy set, with  $\text{dom}(F) = \{l_1, \dots, l_n\}$ , and  $F_{-k}$  the H-IFS resulting from the restriction of  $F$  to the domain  $\text{dom}(F) \setminus \{l_k\}$ .  $l_k$  is deducible in  $F$  if

$$\langle \mu, \nu \rangle_{\text{clos}(F_{-k})(l_k)} = \langle \mu, \nu \rangle_{\text{clos}(F)(l_k)}$$

As a first intuition, it can be said that removing a derivable element from a hierarchical fuzzy set allows one to eliminate redundant information. But, an element being derivable in  $F$  does not necessarily mean that removing it from  $F$  will have no consequence on the closure: removing  $k$  from  $F$  will not impact the degree associated with  $k$  itself in the closure, but it may impact the degrees of the sub-elements of  $k$  in the closure.

For instance, if the element Pasteurised milk is derivable in  $Q$ , according to Definition 4.3.2, removing Pasteurised-milk  $\langle 1, 0 \rangle$  from  $Q$  would not modify the degree of Pasteurised milk itself in the resulting closure, but it could modify the degree of its sub-element Whole-pasteurised-milk. Thus, Pasteurised-milk  $\langle 1, 0 \rangle$  can not be derived or removed. This remark leads us to the following definition of a minimal hierarchical fuzzy set.

**Definition 4.3.3.** In a given equivalence class (that is, for a given closure  $C$ ), a hierarchical fuzzy set is said to be minimal if its closure is  $C$  and if none of the elements of its domain is derivable.

For instance the H-IFSs  $S_1$  and  $S_2$  are *minimal* (none of their elements is derivable). They cannot be reduced further.

$$S_1 = \text{Milk} \langle 1, 0 \rangle$$

$$S_2 = \{ \text{Milk} \langle 1, 0 \rangle, \text{Whole-Milk} \langle 0.7, 0.1 \rangle, \text{Whole-Pasteurised-milk} \langle 1, 0 \rangle, \text{Condensed-Milk} \langle 0.4, 0.3 \rangle \}$$

## 5 Replacing and Constraining Unknown Attribute Values

All descendents of an instance of a high-level concept are replaced with a *minimal H-IFS* has these descendents as members. A *null* value is regarded as a partial value with all base domain values as members. We refer to the

resultant partial value, obtained as a result of this process, as a *primal* partial value. The replacement process is thus performed by the following procedure:

---

Procedure: replacement

---

*Input:* A concept table R consisting of partial values, or nulls

*Output:* A re-engineered partial value table U

*Method:* For each attribute value of R recursively replace the cell value by a *primal* partial value. For each cell of R replace, the primal partial value by a pruned prime-partial -value, until a minimal partial value is reached

---

If a particular member of a partial value violates the domain constraint (rule) then it is *pruned* from the *minimal H-IFS primal partial value*. This process is continued until all partial values have been pruned by the constraints as much as possible. We refer to the resultant partial value, obtained as a result of this process, as a *minimal* partial value.

In addition in an integrated DBMS environment it will be also useful not to query all sources, but only those that contain information relevant to our request. This is quite critical for achieving better query performance. For this reason we equip our Integrated architecture with a repository that contains various constraints (i.e. Intuitionistic Fuzzy Range Constraints, Intuitionistic Fuzzy Functional Dependencies, etc) that are related to the information sources that participate in the Integrated Architecture.

*Range constraints:* such as “The average income per person is estimated to be in the range of €50K”. Considering a finite universe of discourse, say X whose cardinality is N. Let us suppose that  $X = \{X_1, X_2, \dots, X_n\}$  and the Intuitionistic fuzzy number  $\sim a$  given by  $\sim a = \{(x_i, \mu_i, \nu_i) : x_i \in X, i = 1, 2, \dots, N\}$  We can express the above constraint as follows  $\sim Income50K \{(49, .8, .1), (50, .9, .02)(51, .7, .15)\}$ .

*Classical data integrity constraints* such as “All persons stored at a source have a unique identifier”.

*Functional Dependencies:* for instance, a source relation S1 (Name, lives, income, Occupation) has a functional dependency  $Name \rightarrow (Lives, \sim Income)$ . These constraints are very useful to compute answers to queries.

There are several reasons we want to consider constraints separately from the query language. Describing constraints separately from the query language can allow us to do reasoning about the usefulness of a data source with respect to a valid user request.

Some of source constraints can be naturally represented as local constraints. Each *local constraint* is defined on one data source only. These constraints carry a rich set of semantics, which can be utilized in query processing. Any projected database instance of source, these conditions must be satisfied by the tuples in the database.

**Definition 5.1.** Let  $si, \dots, sl$  be  $l$  sources in a data-integrated system. Let  $P = \{pi, \dots, pn\}$  be a set of global predicates, on which the contents of each

source  $s$  are defined. A general global constraint is a condition that should be satisfied by any database instance of the global predicates  $P$ .

General global constraints can be introduced during the design phase of such a data-integration system. That is, even if new sources join or existing ones leave the system, it is assumed that these constraints should be satisfied by any database instance of the global predicates. Given the global predicate *Income*, if a query asks for citizens with an average income above  $\sim \text{Income}60K$ , without checking the source contents and constraints, the integrated system can immediately know that the answer is empty.

To this extent we can interrogate the constraints repository to find out if a particular source contains relevant information with respect to particular request. We now consider the problem of aggregation for the partial value data model. In what follows we are concerned with symbolic attributes, which are typically described by counts and summarised by aggregated tables. The objective is to provide an aggregation operator which allows us to aggregate individual tuples to form summary tables.

## 6 Summary Tables and Aggregation

A summary table  $R$ , is represented in the form [23] of an Intuitionistic fuzzy relation (IFR).

*Aggregation (A):* An aggregation operator  $A$  is a function  $A(G)$  where  $G = \{ \langle x, \mu_F(x), \nu_F(x) \rangle \mid x \in X \}$  where  $x = \langle att_1, \dots, att_n \rangle$  is an ordered tuple belonging to a given universe  $X$ ,  $\{att_1, \dots, att_n\}$  is the set of attributes of the elements of  $X$ ,  $\mu_F(x)$  and  $\nu_F(x)$  are the degree of membership and non-membership of  $x$ . The result is a bag of the type  $\{ \langle x', \mu_F(x'), \nu_F(x') \rangle \mid x' \in X \}$ . To this extent, the bag is a group of elements that can be duplicated and each one has a degree of  $\mu$  and  $\nu$ .

*Input:*  $R_i = (l, F, H)$  and the function  $A(G)$

*Output:*  $R_o = (l_o, F_o, H_o)$  where

- $l$  is a set of levels  $l_1, \dots, l_n$ , that belong to a partial order  $\leq O$

To identify the level  $l$  as part of a hierarchy we use  $dl$ .

$$l_{\perp} : \text{base level} \quad l_{\top} : \text{top level}$$

for each pair of levels  $l_i$  and  $l_j$  we have the relation

$$\mu_{ij} : l_i \times l_j \rightarrow [0, 1] \quad \nu_{ij} : l_i \times l_j \rightarrow [0, 1] \quad 0 < \mu_{ij} + \nu_{ij} < 1$$

- $F$  is a set of fact instances with schema  $F = \{ \langle x, \mu_F(x), \nu_F(x) \rangle \mid x \in X \}$ , where  $x = \langle att_1, \dots, att_n \rangle$  is an ordered tuple belonging to a given universe  $X$ ,  $\mu_F(x)$  and  $\nu_F(x)$  are the degree of membership and non-membership of  $x$  in the fact table  $F$  respectively.

- $H$  is an object type history that corresponds to a structure  $(l, F, H')$  which allows us to trace back the evolution of a structure after performing a set of operators i.e. aggregation

The definition of the extended group operators allows us to define the extended group operators *Roll up* ( $\Delta$ ), and *Roll Down* ( $\Omega$ ).

*Roll up* ( $\Delta$ ): The result of applying Roll up over dimension  $d_i$  at level  $dl_i$  using the aggregation operator  $A$  over a relation  $R_i = (l_i, F_i, H_i)$  is another relation  $R_o = (l_o, F_o, H_o)$

*Input:*  $R_i = (l_i, F_i, H_i)$

*Output:*  $R_o = (l_o, F_o, H_o)$

An object of type history is a recursive structure  $H = \begin{cases} \omega \text{ is the initial state} \\ \text{of the relation.} \\ (l, A, H') \text{ is the state} \\ \text{of the relation after} \\ \text{performing an} \\ \text{operation on it.} \end{cases}$

The structured history of the relation allows us to keep all the information when applying *Roll up* and get it all back when *Roll Down* is performed. To be able to apply the operation of *Roll Up* we need to make use of the  $IF_{SUM}$  aggregation operator.

*Roll Down* ( $\Omega$ ): This operator performs the opposite function of the *Roll Up* operator. It is used to roll down from the higher levels of the hierarchy with a greater degree of generalization, to the leaves with the greater degree of precision. The result of applying *Roll Down* over a relation  $R_i = (l, F, H)$  having  $H = (l', A', H')$  is another relation  $R_o = (l', F', H')$ .

*Input:*  $R_i = (l, F, H)$

*Output:*  $R_o = (l', F', H')$  where  $F' \rightarrow \text{set of fact instances defined by operator } A$ .

To this extent, the *Roll Down* operative makes use of the recursive history structure previously created after performing the *Roll Up* operator.

The definition of aggregation operator points to the need of defining the IF extensions for traditional group operators [20], such as  $SUM$ ,  $AVG$ ,  $MIN$  and  $MAX$ . Based on the standard group operators, we provide their IF extensions and meaning.

$IF_{SUM}$ : The  $IF_{sum}$  aggregate, like its standard counterpart, is only defined for numeric domains. The relation  $R$  consists of tuples  $R_i$  with  $1 \leq i \leq m$ . The tuples  $R_i$  are assumed to take Intuitionistic Fuzzy values for the attribute  $att_{n-1}$  for  $i = 1$  to  $m$  we have  $R_i[att_{n-1}] = \{ \langle \mu_i(u_{ki}), \nu_i(u_{ki}) \rangle / u_{ki} | 1 \leq k_i \leq n \}$ . The  $IF_{sum}$  of the attribute  $att_{n-1}$  of the relation  $R$  is defined by:

$$IF_{SUM}((att_{n-1})(R)) = \left\{ \langle u \rangle / y \mid \left( \left( u = \min_{i=1}^m (\mu_i(u_{ki}), \nu_i(u_{ki})) \right) \wedge \left( y = \sum_{k_i=k_1}^{km} u_{ki} \right) \left( \forall_{k_1, \dots, km} : 1 \leq k_1, \dots, km \leq n \right) \right) \right\}$$

*Example:*

$$\begin{aligned}
& IF_{SUM}((Amount)(ProdID)) \\
&= \{ \langle .8, .1 \rangle / 10 \} + \{ \langle .4, .2 \rangle / 11, \langle .3, .2 \rangle / 12 \} \\
&\quad + \{ \langle .5, .3 \rangle / 13, \langle .5, .1 \rangle / 12 \} \\
&= \{ \langle .8 \wedge .4, .1 \wedge .2 \rangle / 10 + 11, \langle .8 \wedge .3, .1 \wedge .2 \rangle / 10 + 12 \} \\
&\quad + \{ \langle .5, .3 \rangle / 13, \langle .5, .1 \rangle / 12 \} \\
&= \{ \langle .4, .2 \rangle / 21, \langle .3, .2 \rangle / 22 \} + \{ \langle .5, .3 \rangle / 13, \langle .5, .1 \rangle / 12 \} \\
&= \{ \langle .4 \wedge .5, .2 \wedge .3 \rangle / 21 + 13, \langle .4 \wedge .5, .2 \wedge .1 \rangle / 21 + 12, \\
&\quad \langle .3 \wedge .5, .2 \wedge .3 \rangle / 22 + 13, \langle .3 \wedge .5, .2 \wedge .1 \rangle / 22 + 12 \} \\
&= \{ \langle .4, .3 \rangle / 34, \langle .4, .2 \rangle / 33, \langle .3, .3 \rangle / 35, \langle .3, .2 \rangle / 34 \} \\
&= \{ \langle .3, .3 \rangle / 34, \langle .4, .2 \rangle / 33, \langle .3, .3 \rangle / 35 \}
\end{aligned}$$

$IF_{AVG}$ : The  $IF_{AVG}$  aggregate, like its standard counterpart, is only defined for numeric domains. This aggregate makes use of the  $IF_{SUM}$  that was discussed previously and the standard  $COUNT$ . The  $IF_{AVG}$  can be defined as:

$$IF_{AVG}((att_{n-1})(R)) = IF_{SUM}((att_{n-1})(R)) / COUNT((att_{n-1})(R))$$

$IF_{MAX}$ : The  $IF_{MAX}$  aggregate, like its standard counterpart, is only defined for numeric domains. The  $IF_{sum}$  of the attribute  $att_{n-1}$  of the relation  $R$  is defined by:

$$\begin{aligned}
& IF_{MAX}((att_{n-1})(R)) \\
&= \{ \langle u \rangle y \mid ((u = \min_{i=1}^m (\mu_i(u_{ki}), \nu_i(u_{ki})) \wedge (y = \max_{i=1}^m (\mu_i(u_{ki}), \nu_i(u_{ki}))) \\
&\quad (\forall k_1, \dots, k_m : 1 \leq k_1, \dots, k_m \leq n)) \}
\end{aligned}$$

$IF_{MIN}$ : The  $IF_{MIN}$  aggregate, like its standard counterpart, is only defined for numeric domains. Given a relation  $R$  defined on the schema  $X$  ( $att_1, \dots, att_n$ ), let  $att_{n-1}$  defined on the domain  $U = \{u_1, \dots, u_n\}$ . The relation  $R$  consists of tuples  $R_i$  with  $1 \leq i \leq m$ . Tuples  $R_i$  are assumed to take Intuitionistic Fuzzy values for the attribute  $att_{n-1}$  for  $i = 1$  to  $m$  we have  $R_i[att_{n-1}] = \{ \langle \mu_i(u_{ki}), \nu_i(u_{ki}) \rangle / u_{ki} \mid 1 \leq k_i \leq n \}$ . The  $IF_{sum}$  of the attribute  $att_{n-1}$  of the relation  $R$  is defined by:

$$\begin{aligned}
& IF_{MIN}((att_{n-1})(R)) \\
&= \{ \langle u \rangle / y \mid ((u = \min_{i=1}^m (\mu_i(u_{ki}), \nu_i(u_{ki})) \wedge (y = \min_{i=1}^m (\mu_i(u_{ki}), \nu_i(u_{ki}))) \\
&\quad (\forall k_1, \dots, k_m : 1 \leq k_1, \dots, k_m \leq n)) \}
\end{aligned}$$

We can observe that the  $IF_{MIN}$  is extended in the same manner as  $IF_{MAX}$  aggregate except for replacing the symbol *max* in the  $IF_{MAX}$  definition with *min*.



6.1 Summarisation Paths

The structure of any H-IFS can be described by a domain concept relation  $DCR = (Concept, Element)$ , where each tuple describes a relation between elements of the domain on different levels.

The DCR can be used in calculating recursively [24] the different summarisation or selection paths as follows:

$$PATH \leftarrow DCR \bigwedge_{\{x=1 \dots (n-2) | n > 2\}} DCR_x$$

If  $n \leq 2$ , then DCR becomes the Path table as it describes all summarisation and selection paths.

These are entries to a knowledge table that holds the metadata on parent-child relationships. An example is presented below:

Figure 5 shows how our Milk hierarchy knowledge table is kept. Paths are created by running a recursive query that reflects the ‘PATH’ algebraic statement. The hierarchical IFS used as example throughout this paper comprises of three levels, thus calling for the SQL-like query as below:

```
SELECT A.Concept as Grand-concept, b.concept, b.element
FROM DCR as A, DCR as B
WHERE A.child = B.parent;
```

This query will produce the following paths (Fig. 6):

DCR	
Concept	Element
Milk <1.0, 0.0>	Pasteurised Milk <1.0, 0.0>
Milk <1.0, 0.0>	Whole Milk <0.7, 0.1>
Milk <1.0, 0.0>	Condensed Milk <0.4, 0.3>
Pasteurised Milk <1.0, 0.0>	Whole Pasteurised Milk <1.0, 0.0>
Whole Milk <0.7, 0.1>	Whole Pasteurised Milk <1.0, 0.0>
Whole Milk <0.7, 0.1>	Whole Condensed Milk <0.7, 0.1>
Condensed Milk <0.4, 0.3>	Whole Condensed Milk <0.7, 0.1>

Fig. 5. Domain concept relation

Path			
Grand-concept	Concept	Element	Path Colour
Milk <1.0, 0.0>	Pasteurised Milk <1.0, 0.0>	Whole Pasteurised Milk <1.0, 0.0>	Red
Milk <1.0, 0.0>	Whole Milk <0.7, 0.1>	Whole Pasteurised Milk <1.0, 0.0>	Blue
Milk <1.0, 0.0>	Whole Milk <0.7, 0.1>	Whole Condensed Milk <0.7, 0.1>	Green
Milk <1.0, 0.0>	Condensed Milk <1.0, 0.0>	Whole Condensed Milk <0.7, 0.1>	Brown

Fig. 6. Path table

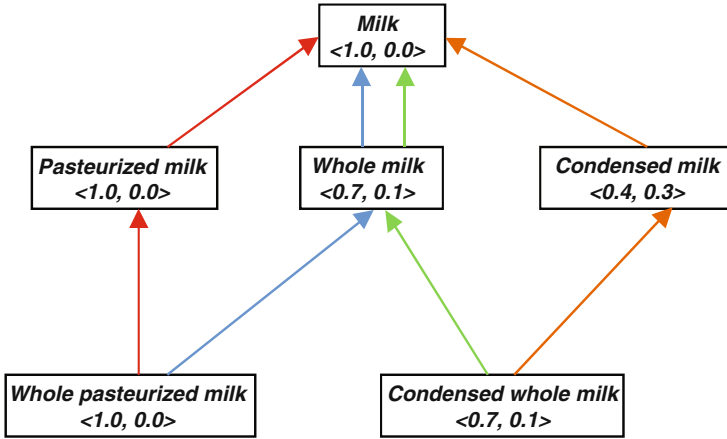


Fig. 7. Pictorial representation of paths

Figure 7 presents a pictorial view of the four distinct summarisation and selection paths.

These paths will be used in fuzzy queries to extract answers that could be either definite or possible. This will be realised with the aid of the predicate ( $\theta$ ).

A predicate ( $\theta$ ) involves a set of atomic predicates ( $\theta_1, \dots, \theta_n$ ) associated with the aid of logical operators  $p$  (i.e.  $\wedge, \vee$ , etc.). Consider a predicate  $\theta$  that takes the value “Whole Milk”,  $\theta$  = “Whole Milk”.

After utilizing the IFS hierarchy presented in Fig. 7, this predicate can be reconstructed as follows:

$$\theta = \theta_1 \vee \theta_2 \vee \dots \vee \theta_n$$

In our example,  $\theta_1$  = “Whole Milk”,  $\theta_2$  = “Whole Pasteurised Milk” and  $\theta_n$  = “Condensed Whole Milk”.

The reconstructed predicate  $\theta = (\text{Whole Milk} \vee \text{Whole Pasteurised Milk} \vee \text{Condensed Whole Milk})$  allows the query mechanism to not only definite answers, but also possible answers [25].

In terms a query retrieving data from a summary table, the output contains not only records that match the initial condition, but also those that satisfy the reconstructed predicate. Consider the case where no records satisfy the initial condition (Whole Milk). Traditional aggregation query would have returned no answer, however, based on our approach, the extended query would even in this case, return an answer, though only a possible one, with a specific belief and disbelief  $\langle \mu, \nu \rangle$ . It will point to those records that satisfy the reconstructed predicate  $\theta$ , more specifically, “Whole Pasteurised Milk and Condensed Whole Milk”.

## 7 Conclusions

We provide a means of using background knowledge to re-engineer the data representation into a partial value representation with the aid of H-IFS and Intuitionistic Fuzzy relational representation.

The hierarchical links are defined by the “kind of,  $\leq$ ” relation. The membership of an element in a H-IFS has consequences on the membership and non-membership of its sub elements in this set. The notion of H-IFS, that may be defined on a part of a hierarchy and the notion of closure of a H-IFS, that is explicitly defined on the whole hierarchy, using the links between the elements that compose the hierarchy.

H-IFSs that have the same closure define equivalence classes, called minimal H-IFS. Minimal fuzzy sets are used as a basis to define the generalization of a H-IFS fuzzy set. The proposed methodology aims at enlarging the user preferences expressed when defining a query, in order to obtain related and complementary answers.

We have discussed how domain knowledge presented in the form of background knowledge, such as integrity constraints, functional dependencies or details of the concept hierarchy, may be used to reduce the amount of missing data in the database.

We have presented a new multidimensional model that is able to operate over data with imprecision in the facts and the summarisation hierarchies. Classical models imposed a rigid structure that made the models present difficulties when merging information from different but still reconcilable sources.

This is likely to be a useful tool for decision support and knowledge discovery in, for example, data mediators, data warehouses, where the data are often subject to such imperfections. Furthermore we notice that our approach can be used for the representation of Intuitionistic fuzzy linguistic terms.

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